AN EXPLICIT EXAMPLE OF KOSZUL DUALITY

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Abstract. The aim of this note is provide an explicit example of Koszul duality between augmented commutative differential graded algebras and L_∞-algebras. We analyze the formal neighborhood of a distinguished point in the derived zero locus of a function $f : \mathbb{A}^1 \to \mathbb{A}^1$. We show this data can be expressed in two Koszul dual ways: as a complete commutative differential graded algebra (cdga) of functions on this formal neighborhood and as a L_∞-algebra structure on the shifted tangent space at the distinguished point.

Contents

1. Functions on a Derived Zero Locus

Consider $\mathbb{A}^1 = \mathrm{Spec}(k[x])$ where k is a field of characteristic 0. Given a map of schemes $f : \mathbb{A}^1 \to \mathbb{A}^1$, we can consider the derived zero locus of f

$$f^{-1}(0) := \mathbb{A}^1 \overset{h}{\underset{\mathbb{A}^1}{\times}} \mathrm{pt}\,.$$

Here the derived intersection is formed by $f : \mathbb{A}^1 \to \mathbb{A}^1$ and the inclusion of the origin $\mathrm{pt} \to \mathbb{A}^1$.

The commutative differential graded algebra (cdga) of functions on $f^{-1}(0)$, denoted $\mathcal{O}_{f^{-1}(0)}$, is given by

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$$\begin{aligned}
\mathcal{O}_{f^{-1}(0)} &\simeq \mathcal{O}_{(\mathbb{A}^1 \overset{h}{\underset{\mathbb{A}^1}{\times}} \mathrm{pt})} \\
&\simeq \mathcal{O}_{\mathbb{A}^1} \overset{\mathbb{L}}{\underset{\mathcal{O}_{\mathbb{A}^1}}{\otimes}} \mathcal{O}_{\mathrm{pt}} \\
&\simeq k[x] \overset{\mathbb{L}}{\underset{k[x]}{\otimes}} k \\
&\simeq k[x] \underset{k[x]}{\otimes} k[x,y] \quad \text{with} \quad \deg(x) = 0, \deg(y) = -1, dy = x \\
&\simeq k[x,y] \quad \text{with} \quad \deg(x) = 0, \deg(y) = -1, dy = f(x).
\end{aligned}$$

Notice that we use cohomological grading conventions. The last algebra in this chain of weak equivalences will be the main focus of this note.

Definition 1.1. Functions on the derived zero locus of f is

$$\mathcal{O}_{f^{-1}(0)} \simeq k[x,y] \quad \text{with} \quad \deg(x) = 0, \deg(y) = -1, dy = f(x).$$

We will now assume without loss of generality that $f(0) = 0$ or equivalently $0 \in f^{-1}(0)$. This distinguished point gives an augmentation of the derived zero locus $\mathcal{O}_{f^{-1}(0)} \to k$. We wish to complete the derived zero locus at this distinguished point. We do this by specifying the complete differential graded commutative algebra of functions on such a completed space.

Definition 1.2. The commutative differential graded algebra of functions on $f^{-1}(0)$ completed at $0 \in \mathbb{A}^1$ is $k[[x]][y]$ with $\deg(x) = 0$, $\deg(y) = -1$, and

$$dy = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

The purpose of this note is to show how to recover this completed space from the data of an L_∞-algebra.

2. Cotangent and Tanget Complex

We now wish to compute to cotangent complex of $\mathcal{O}_{f^{-1}(0)}$ and the derived cotangent space of $\mathcal{O}_{f^{-1}(0)}$ at the point $0 \in \mathbb{A}^1$. There are many beautiful ways to think abstractly about the cotangent complex functor and how it relates to the stabilization of a homotopical category differential graded commutative algebras. However, in these notes we will focus on a computation model for the cotangent complex.

Definition 2.1. Let A be a cdga, then the A-module of Kahler differentials on A is $\Omega_{A/k} := F_A/I$ where

- $F_A := \bigoplus A \, d_{\mathrm{dR}} \, x$, where $x \in A$ is a homogeneous element and $d_{\mathrm{dR}} \, x$ is new variable associated to x of the same degree. The differential on F_A is given by

$$d(ad_{dR}x) := da \, d_{\mathrm{dR}} \, x + (-1)^{\bar{a}} a \, d_{\mathrm{dR}}(dx).$$

- $I \subset F_A$ is the ideal generated by the equations

$$d_{\mathrm{dR}}(x+y) = d_{\mathrm{dR}} \, x + d_{\mathrm{dR}} \, y, \quad d_{\mathrm{dR}}(xy) = y \, d_{\mathrm{dR}} \, x + (-1)^{\bar{x}\bar{y}} x \, d_{\mathrm{dR}} \, y, \quad d_{\mathrm{dR}}(b) = 0 \text{ for } b \in k.$$

The cotangent complex of a cdga A is a derived version of the A-module of Kahler differentials defined above. What do mean by the word "derived"? We mean that we replace the cdga A by another cdga R quasi-isomorphic A, where R is constructed specifically to have certain properties. The properties are chosen so that the functor of constructing Kahler differentials preserves quasi-isomorphisms between all cdgas with these properties. The necessity for finding a replacement with these properties follows from the existence of two quasi-isomorphic cdgas whose module of Kahler differentials are not quasi-isomorphic. It is an exercise to find two such algebras. In summary, the functor of forming the module of Kahler differentials does not preserve quasi-isomorphisms in general.

The type of resolution we seek is called a semi-free resolution of a cdga A.

Definition 2.2. A cdga (R, d) is called semi-free if

- The underlying graded algebra $R^{\#}$ is a polynomial algebra.
- There is an exhaustive filtration $0 = F^0 \subseteq F^1 \subseteq \ldots \subseteq R$ with $d(F^n) \subseteq F^{n-1}$.

Again, the functor of forming Kahler differentials preserves quasi-isomorphisms between two semi-free cdgas.

Theorem 2.3. *For every cdga A, there exist a semi-free cdga R with a quasi-ismorphism $R \to A$.*

We are now in position to define the cotangent complex.

Definition 2.4. The cotangent complex of a cdga A is the A-module

$$L_A := \Omega_{R/k} \otimes_R A$$

where $R \to A$ is a semi-free resolution of A. Furthermore, given an augmentation $A \to k$, we can define the derived cotangent space at the point $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ by

$$k \otimes_A L_A.$$

Definition 2.5. The tangent complex of a cdga A is the A-module

$$L_A^{\vee}$$

where we take the dual in the category of (dg) A-modules. Similarly, we can define the derived tangent space at the point $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ by

$$T := (k \otimes_A L_A)^\vee$$

where we take the dual in the category of (dg) k-modules.

We now work through all of the definitions above in our example where the cdga $A = k[x, y]$ with differential $dy = f(x)$.

We are lucky! This algebra is already semi-free. The filtration $0 \subset k[x] \subset k[x, y]$ is indeed exhaustive and since $dx = 0$ and $dy = f(x)$, the differential satisfies the required property with respect to this filtration. Thus, to compute the cotangent complex we need only compute the module of Kahler differentials. Using the notation in the definition for Kahler differentials we have $\Omega_{A/k} = A\,\mathrm{d_{dR}}\,x \oplus A\,\mathrm{d_{dR}}\,y$ with differential $d(\mathrm{d_{dR}}\,y) = \frac{\partial f}{\partial x}\,\mathrm{d_{dR}}\,x$. So we have

Proposition 2.6. *The cotangent complex of $A = \mathcal{O}_{f^{-1}(0)}$ is $A\,\mathrm{d_{dR}}\,x \oplus A\,\mathrm{d_{dR}}\,y$ with differential $d(\mathrm{d_{dR}}\,y) = \frac{\partial f}{\partial x}\,\mathrm{d_{dR}}\,x$. The cotangent space of $A = \mathcal{O}_{f^{-1}(0)}$ at the point $0 \in \mathbb{A}^1$ is $k\,\mathrm{d_{dR}}\,x \oplus k\,\mathrm{d_{dR}}\,y$ with differential $d(\mathrm{d_{dR}}\,y) = f'(0)\,\mathrm{d_{dR}}\,x$.*

Notation 2.7. In the sequel we will refer to the cotangent space of $A = \mathcal{O}_{f^{-1}(0)}$ at the point $0 \in \mathbb{A}^1$ as the dg vector space $k \oplus k[1]$. The first summand corresponds to $k\,\mathrm{d_{dR}}\,x$ and the second to $k\,\mathrm{d_{dR}}\,y$. Recall the notation $k[1]$ means place the 1-dimension vector space $k\,\mathrm{d_{dR}}\,y$ in cohomological degree -1.

Taking the dual of $k \oplus k[1]$ we get:

Proposition 2.8. *The derived tangent space of $A = \mathcal{O}_{f^{-1}(0)}$ at the point $0 \in \mathbb{A}^1$ is*

$$T \simeq k \oplus k[-1].$$

The shifted derived tangent space is

$$T[-1] \simeq k[-1] \oplus k[-2].$$

We want to put an L_∞-algebra structure on the shifted derived tangent space, $T[-1]$, but first we must review the definition of an L_∞-algebra.

3. The definition of an L_∞-Algebra

An L_∞-algebra is a homotopical version of a Lie Algebra. Instead of a having bracket that satisfies the Jacobi identity, we only ask the Jacobi identity hold up to homotopy. The precise definition is as follows.

Definition 3.1. An L_∞-algebra is a graded vector space $\mathfrak{g}$ with a sequence of graded antisymmetric k-brackets $[-, -, \ldots, -] =: l_k(-, -, \ldots, -)$ for every $k > 0$. These

k-brackets are maps $l_k : \wedge^k \mathfrak{g}[k-2] \to \mathfrak{g}$ of degree 0. The k-brackets are asked to satisfy the higher Jacobi relations

$$\sum_{i=1}^{n}(-1)^k \sum_{\substack{i_1<\ldots<i_k;\ j_1<\ldots j_{n-k} \\ \{i_1,\ldots i_k\}\cup\{j_1,\ldots j_{n-k}\}=\{1,\ldots,n\}}} (-1)^\epsilon [[x_{i_1},\ldots,x_{i_k}],x_{j_1},\ldots,x_{j_{n-k}}] = 0$$

Here the sign $(-1)^\epsilon$ equals the product of the sign $(-1)^\pi$ associated to the permutation

$$\pi = \begin{pmatrix} 1\ldots k & k+1\ldots n \\ i_1\ldots i_k & j_1\ldots j_{n-k} \end{pmatrix}$$

with the sign associated by the Koszul sign convention to the action of π on the elements $(x_1,\ldots,x_n)$ of $\mathfrak{g}$.

Remark 3.2. Let's unpack the definition above a little bit.

- When $n=1$, we have a map $l_1 : \mathfrak{g}[-1] \to \mathfrak{g}$ and the higher Jacobi relation gives $(l_1)^2 = 0$. We denote $\partial := l_1$.
- When $n=2$, we have a map $[-,-] : \wedge^2 \mathfrak{g} \to \mathfrak{g}$ and the higher Jacobi relation gives
$$-[\partial x_1, x_2] - (-1)^{\bar{x_1}\bar{x_2}+1}[\partial x_2, x_1] + \partial[x_1,x_2] = 0.$$
or equivalently
$$\partial[x_1,x_2] = [\partial x_1, x_2] + (-1)^{\bar{x_1}}[x_1, \partial x_2].$$
This says that ∂ is a derivation for the bracket $[-,-]$.
- When $n=3$, we have a map $[-,-,-] : \wedge^3 \mathfrak{g}[1] \to \mathfrak{g}$ and the higher Jacobi relation gives

$$\begin{aligned}
&-[\partial x_1, x_2, x_3] - (-1)^{\bar{x_1}\bar{x_2}+1}[\partial x_2, x_1, x_3] - (-1)^{\bar{x_3}(\bar{x_1}+\bar{x_2})+2}[\partial x_3, x_1, x_2] \\
&+[[x_1,x_2],x_3] + (-1)^{\bar{x_1}(\bar{x_2}+\bar{x_3})+2}[[x_2,x_3],x_1] + (-1)^{\bar{x_2}\bar{x_3}+1}[[x_1,x_3],x_2] \\
&-\partial[x_1,x_2,x_3] = 0
\end{aligned}$$

or equivalently

$$\begin{aligned}
&[[x_1,x_2],x_3] + (-1)^{\bar{x_1}(\bar{x_2}+\bar{x_3})}[[x_2,x_3],x_1] + (-1)^{\bar{x_2}\bar{x_1}+1}[[x_1,x_3],x_2] = \\
&\partial[x_1,x_2,x_3] + [\partial x_1, x_2, x_3] + (-1)^{\bar{x_1}\bar{x_2}+1}[\partial x_2, x_1, x_3] + (-1)^{\bar{x_3}(\bar{x_1}+\bar{x_2})}[\partial x_3, x_1, x_2]
\end{aligned}$$

which reduces to

$$\begin{aligned}
&[[x_1,x_2],x_3] + (-1)^{\bar{x_1}(\bar{x_2}+\bar{x_3})}[[x_2,x_3],x_1] + (-1)^{\bar{x_3}(\bar{x_1}+\bar{x_2})}[[x_3,x_1],x_2] = \\
&\partial[x_1,x_2,x_3] + [\partial x_1, x_2, x_3] + (-1)^{\bar{x_1}}[x_1, \partial x_2, x_3] + (-1)^{\bar{x_1}+\bar{x_2}}[x_1, x_2, \partial x_3].
\end{aligned}$$

Thus, if the 3-bracket $l_3 = 0$, then we recover the usual graded Jacobi identity. If in addition, all k-brackets for $k \geq 3$ are trivial, then the data of an L_∞-algebra reduces to the data of a dg Lie algebra.

Remark 3.3. If you like operads and model categories, then you know there is a model category of operads. In this language, the L_∞ operad is a cofibrant replacement of the Lie operad in the model category of operads. More generally, the "∞"-version of any operad is formed by taking its cofibrant replacement in the model category of operads.

The following is an equivalent definition of an L_∞-algebra.

Definition 3.4. An L_∞-algebra is a graded vector space $\mathfrak{g}$ with a degree 1 derivation

$$d : \widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1]) \to \widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1])$$

satisfying

- $d^2 = 0$.
- d makes $\widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1])$ into a cdga over the field k.

We will call the cdga $\widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1])$ the Chevalley-Eilenberg cochain complex of the L_∞-algebra $\mathfrak{g}$. The cohomology with respect to the differential d is called the Chevalley-Eilenberg cohomology of the L_∞-algebra $\mathfrak{g}$.

Remark 3.5. Starting with the data $(\widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1]), d)$, we can recover the antisymmetric n-brackets on $\mathfrak{g}$. Namely, since d is a derivation, it is completely determined by a linear map $d : \mathfrak{g}^\vee[-1] \to \widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1])$. Decomposing the target space we get a sequence of maps $d_k : \mathfrak{g}^\vee[-1] \to \mathrm{Sym}^k(\mathfrak{g}^\vee[-1])$. If we take the dual of each map in this sequence, we get another sequence of maps $\mathrm{Sym}^k(\mathfrak{g}^\vee[-1])^\vee \to (\mathfrak{g}^\vee[-1])^\vee$. This is equivalent to a sequence of maps of degree 0 maps $(\wedge\, \mathfrak{g})^k[k-2] \to \mathfrak{g}$, which are k-brackets on $\mathfrak{g}$ used in the first definition of an L_∞-algebra. The higher Jacobi relations are encoded by the surprisingly simple formula $d^2 = 0$.

Remark 3.6. By Koszul duality, we mean the process of moving between an L_∞-algebra and a cdga.

4. L_∞-Structure on the Shifted Tangent Space

We now discuss the L_∞-structure on the shifted tangent space

$$\mathfrak{g} := k[-1] \oplus k[-2]$$

at the point 0 in the derived zero locus of $f : \mathbb{A}^1 \to \mathbb{A}^1$. That is, we must specify a sequence of maps $l_k : \wedge^k \mathfrak{g}[k-2] \to \mathfrak{g}$ of degree 0. Let us examine the structure of these maps in this example.

- If $k = 0$, we have a map $l_0 : k[-2] \to \mathfrak{g}$. That is, a map
$$k[-2] \to k[-1] \oplus k[-2].$$
- If $k = 1$, we have a map $l_1 : \mathfrak{g}[-1] \to \mathfrak{g}$. That is, a map
$$k[-2] \oplus k[-3] \to k[-1] \oplus k[-2].$$

- If $k = 2$, we have a map $l_2 : \wedge^2 \mathfrak{g} \to \mathfrak{g}$. That is, a map

$$(k[-1] \oplus k[-2]) \wedge (k[-1] \oplus k[-2]) \to k[-1] \oplus k[-2].$$

- If $k = n$, we have a map $l_n : \wedge^n \mathfrak{g}[n-2] \to \mathfrak{g}$.

The main observation is that since every l_n is a map of degree 0, it is completely determined by a map $k[-2] \to k[-2]$. Furthermore, a map between 1-dimensional vector spaces is equivalent to the data of a number.

Now consider the Chevalley-Eilenberg cochain complex on $\mathfrak{g} = k[-1] \oplus k[-2]$. Recall this is $(\widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1]), d)$ where $d : \widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1]) \to \widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1])$ is a derivation of degree 1. Furthermore, we can decompose d into pieces $d_k : \mathfrak{g}^\vee[-1] \to \mathrm{Sym}^k(\mathfrak{g}^\vee[-1])$ where, by definition, the pieces d_n are dual to the maps l_n described above. Thus, the pieces d_k are given by numbers.

Now, if $\mathfrak{g} = k[-1] \oplus k[-2]$ then $\mathfrak{g}^\vee[-1] = k \oplus k[1]$. Thus, the maps

$$d_k : \mathfrak{g}^\vee[-1] \to \mathrm{Sym}^k(\mathfrak{g}^\vee[-1])$$

take the form

$$d_k : k \oplus k[1] \to \mathrm{Sym}^k(k \oplus k[1]).$$

where the first and second summands of $k \oplus k[1]$ correspond to variables x and y. Since d is derivation of degree 1, the map d_k is still of degree 1. Hence

$$d_k(y) = a_k x^k$$

for some number a_k in the field k. Thus, our Chevalley-Eilenberg complex on the shifted cotangent space is

$$\widehat{\mathrm{Sym}}(k \oplus k[1]) \simeq k[[x]][y]$$

with

$$dy = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Choosing the following values for the numbers $\{a_0, a_1, a_2, \ldots\}$

- $a_0 = f(0)$
- $a_1 = f'(0)$
- $a_2 = f''(0)/2!$
- $a_n = f^{(n)}(0)/n!$

equips $\mathfrak{g} = k[-1] \oplus k[-2]$ with the structure of an L_∞-algebra.

We have proven that

Proposition 4.1. *The Chevalley-Eilenberg complex of the* L_∞*-algebra* $\mathfrak{g}$ *(the shifted tangent space at* 0 *of the derived zero locus of* $f : \mathbb{A}^1 \to \mathbb{A}^1$*) is equivalent to the cdga of functions on derived zero locus of* f *completed at the point* 0*. In other words, there is an equivalence of cdgas*

$$(\widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1]), d) \simeq k[[x]][y]$$

with $\deg(x) = 0$, $\deg(y) = -1$, *and* $dy = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$

Thus, we have recovered the formal neighborhood the distinguished point inside of the derived zero locus as desired.

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